THE INFINITY OF NATURAL NUMBERS

By Aristo Tacoma, April 12, 2019.

Consider the following notion, as set forth by L.E.J. Brouwer, that mathematics must be led by "clear ideas" throughout.

Now consider my argument of why N={1,2,3,…} where N is, as is typically said, set of all 'natural numbers' of a finite positive kind and only finite numbers, while the set as a whole is infinite. This is a construction that, by a certain type of reductio ad absurdum, is having a self-contradiction in it. For those who have worked with my previous ways of showing this, what I just said has clear meaning. For others, it is surprising, at best, and usually considered highly unlikely.

So, I will now give an argument that, while, as far as I can say, is entirely correct, is not considered relevant in 20th century style of mathematics. For those who are not deeply into this style of mathematics, which is still, after all, fairly dominant so far into the 21st century, it can be very surprising that the argument isn't considered very relevant. I have published this regularly since 2004, in various forms, consult eg avenuege.com/library. To clear up this confusion I have written an addition to this form of the argument, as Part II ("Bird's eye perspective on mathematics").

PART I. Reductio ad absurdum

Definition. N is the set of 1, 2, 3 and upwards, where we add one to the last member added to create the next.
Assumption 1. N has every finite positive whole number in it.
Assumption 2. N has no other members that finite positive whole numbers in it.

Proposition: N exists.
We will no defeat this proposition.

We look at its definition. Using the principle of Brouwer, that we must have clear ideas about each step, let us look at each step.

Step 1: N={1}.
Step 2: N={1,2}
Step 3:N={1,2,3}.

We write these steps as follows, reading from below and up:

1 2 3
1 2
1

We then look for the clear idea in going on with this, first some more finite steps:

1 2 3 4
1 2 3
1 2
1

then

1 2 3 4 5
1 2 3 4
1 2 3
1 2
1

We see here, clearly, that the direction UP represents the idea, 'adding one new member to the set'.
In other words, the height of this triangle as it stands, is longer the more members we add to the set.
We see here also clearly, that uppermost line of the triangle represents a measure of the highest number so far added to the set, in this step by step process.
There is no necessity in using a 10-digit system here. Let us rather write it like this:

Step 1:
I

Step 2:
I I
I

Step 3:
I I I
I I
I

Step 4:

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I I I I
I I I
I I
I
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And indeed, by this the simplest possible representation of numbers, where 'I I I I' can be said to represent the number four, we can see that the uppermost horisontal line of the triangle represents the size of the latest member, while the height of the triangle represents the size of the set, so far in the building process.

In now formatting this with equal spacing vertically and horisontally we will have symmetrical triangles, ie, triangles whose upper horisontal line has equal length, at every step in the process, as the height of the triangle. This, then, is step 5:

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I I I I I
I I I I
I I I
I I
I
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We are now in the position to examine the assumedly clear idea of 'etc' as applied to this process of building the set N, which we have assumed exists. We imagine that the process as here symbolized goes on and on without end. This can be indicated by the following use of dots (again, we should have a certain formatting to see this exactly, monospace etc):

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               .
             .
           .
I I I I I
I I I I
I I I
I I
I
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Give the right monospace formatting, we have three dots going 'up and to the right', to symbolize this notion of going on endlessly. It goes up 45 degrees, indicating that there is a perfect symmetry in this triangle: the upper horisontal line is equal, at every point in the construction of this triangle, to its vertical height line.

The assumption 2 must then be false. For assumption says that the vertical height line is infinitely long while the horisontal line is only finitely long. We conclude that N does not exist.

(This is an argument Brouwer never made, but he hinted at it when he tried to show that R might be countable after all, giving a 'branching tree' method.)

The consequence is (as Brouwer often suggested) that there is an incoherence or unclarity deep within most of foundational mathematics, and proofs, even though rigorously checked by many mathematicians, cannot be trusted. (He applied this even to his own fundamental contributions to topology). Argument in favour of ordinary mathematics then is that 'it seems to work'. This may be because infinity is not actually computed with, merely 'normalized away' or treated from a distance by means of limit. But if there is a real incoherence at the level of ideas, it may mean that it after all can emerge as a self-contradiction at a formal level here and there where it is not expected, just as with a not entirely checked computer program.

PART II: BIRD’S EYE PERSPECTIVE ON MATHEMATICS

It is said that doubt, and sceptical wonder, is the motor of science. While this is a wonderful idea, the statement is often qualified as to mean, ‘at the points where it is interesting to have doubt’. Now what determines this interest? For what falls outside of this interest is considered moot, until, perhaps, a revolution or such in science.

Clearly, what is considered interesting to enquire into, and have doubt about, early in the 21st century is different from eg around 150 years ago. Now, mathematics is a field that has many times as many theorems and proofs and such in it than a single person can comprehend even given a hundred years of full-time study. We can consider mathematics to be a very giant computer program indeed. The program is too complex to allow penetration by a single human mind except as for general principles.

A hundred and fifty years ago, the situation was, as far as I am aware, rather different. If one had gone through such as Euclid's geometry and Descartes' notions on a two-dimension XY coordinate system, one was deep into mathematics and the goal of getting through the rest was at least dimly visible within an active person's career, at least within the Western European universities.

Suppose we have a computer program that, although gigantic, has run perfectly for many decades, and in that process of running, what seems to be the best minds of humanity has declared that the program has no flaws, and indeed it has never shown any flaw, but rather, this program has served many people, even with money. If then somebody points a finger to a funny passage in the code that has never been shown even once to perform with flaws, and that can be argued that it will never perform with flaws, and says: "No, this one lacks clear ideas", it will not be strange if that finger-pointing fails to arouse interest.

But while this metaphor might serve for mathematics as it is now, a hundred and fifty years ago, or so, there was less at stake, because the pathways of reasoning were fewer and less cemented. Today, somebody who insists on clear ideas all the way will never even begin to get through mathematics; then, somebody who insisted could manage to get through it. Today, if somebody says something to the effect that 'formalisms matter more than meaning', he or she is not laughed at. Then, in mid 19th century, it would raise eyebrows.

So what we have is a practical change: the masses of formalisms have lead to 'the formalistic attitude'. Attention has moved from every foundational issue that hasn't been too obvious, and the sense is that 'everything has already been worked with' at all essential points. This despite the fact that among those who are most advanced in working with foundations there are those that do occasionally pronounce that all is not well with the foundations of mathematics. (For instance, such a statement found its way into the description of 'Foundations of Mathematics' in the Enclyclopedica Brittanica as of the 1980s. Whether that statement survived into later editions I have not checked up.)

Now, the formalistic attitude is willing to be what we can call 'impatient' about essential definitions. One who insists on clear ideas is essentially taking a philosophical approach to it all, in which mind rather than formalism must have the upper hand at every point.

Now suppose the vast program (to use that metaphor again) really has an issue, something that will cause the whole program to run wrong given very special conditions indeed--but that these conditions are unknown. However it can be pointed out and explained clearly for those who are patient, so that one can agree in principle. Even then people of a formalistic streak can say: well but it has never failed us so far, so let's just keep on running this program and do more interesting things; what has worked for a century cannot suddenly stop working.

You see that the formalistic attitude also has flavours of the pragmatic attitude: if it is works, it is true; we haven't seen that it doesn't work, so we haven't seen that it is false.

To use a phrase from economics: Mathematics has become 'too big to fail'.

I am not now going to go into the possibility that indeed it has failed in a big way in some areas of the study of physical phenomena where infinities are assumed to be at play, such as as regard what is termed by some 'singularities' in cosmology.

In a nutshell, those who insist on the premise of 'clear ideas' are insisting on allowing scepticism to stop any formulation of something logical, if it doesn't clearly seem to be in every sense logical. They are paying huge attention to the sense of clarity in the mind. They are perceiving how well they are perceiving the idea. They may or may not believe in a kind of 'platonic world' represented by the equations, a field of abstract patterns. But they are believing in the notion that the mind can and must look at what is talked about, forward and backward, from below and from the top, from every angle, and that which doesn't work out as coherent given this approach, is discarded.

It is now very clear, if you have read Part I slowly and carefully, that if you look at the set of all numbers from 1, 2, 3, and up as a whole, it is not only an infinite whole, but a whole that doesn't live up to the assumption that the set has only finite members.

So how can anyone of a formalistic attitude nevertheless believe that N = {1, 2, 3, ...} is a problem-free definition?

The clue is that the formalist deliberately (or as a habit) restricts attention. Attention is restricted to the first part of the construction process. 1. 1+1. 1+1+1. The first part of the construction process has only finite members. They say, if we can reach a finite number this way, it is in the set, and we admit of no more members.

And they give it no more attention. They jump to the next question. If somebody raises a finger and says, "But look at the whole of it! It is infinite! Therefore, by the triangle type of argument we looked at, it cannot only have finite members!"--then they can answer, "Well then don't look at it!"

We who are of a philosophical nature, dedicated to the notion of dialogue and attention to hidden assumptions, dedicated to coherence and not fooling ourselves, may laugh at this rediculous attitude. But let us for a moment try to take it seriously, so we understand it a little bit more.

If you stare at N = {1, 2, 3, ...} there is no glaring inconsistency erupting from the symbols, at least not if you have been conditioned like the typical mathematician of the 20th century or even early 21st century. Also, there is no obvious introduction of anything non-finite. They say, "I can reach any number by this process of adding 1, and those numbers, and only those, are in the set, and they are obviously all finite."

They will then simply look away from the triangle argument and say perhaps roughly something this: "you are attempting to visualize infinity. That is always a confusing thing. Stick to the construction process. Stick to the formalism that defines it. There is no infinity there. Infinity is just a word we use to describe the result--we do not analyze it."

Now I will show how the philosophically inclined person can argue against this person, but I will not guarantee that it actually has any effect on this person. But I believe the argument is important to state, and that I may not have stated it in this way before:

The construction of "N", as outlined above, involves two things: One, the initial construction process. This is, to use a word from computer science, “algorithmic”. One plus one equals two. Two plus one equals three. That’s almost mechanical. It is algorithmic.

So the algorithmic involves what we can call a tight relationship of concepts of a similar kind. Plus. One. Two. Three. Sum. Add one.

But in order to actually define N, we have got to say something more. And that which we then introduce may be said so quickly, and so fast, that we don’t quite notice it, but it is there. Let’s go back and, as an example, spot the change of mental context in this phrase: “I can reach any number by this process of adding 1, and those numbers, and only those, are in the set, and they are obviously all finite.”

Algorthimic phrase: “this process of adding 1”

Change of mental context: “any number”, “all”.

I suggest that what we have here, in addition to the algorithmic process, involves another tightly related set of words, which includes such as “every”, “arbitrary”, “any”, “et cetera”, “whole” and so on.

The definition of N = {1, 2, 3, ...} involves two things:

1. An algorithmic part that gives the start.

2. A different type of part altogether, in which words such as “every”, “any”, or “all” appears, words which, though referring to diferent concepts, refers to tightly related concepts, of a kind that aren’t algorithmic. The word “infinite” also belongs to this group, for there are infinitely many numbers when we speak of ‘every’.

The numbers 1, 2, and 3, represents here the algorithmic part. Even though the “plus 1” is not mentioned explicitly, it is understood implicitly. The “...” represents the non-algorithmic part, namely that we can reach “any” number this way.

If somebody who is of a formalistic streak tries to rescue the definition of N then says, “All right, but I define now N so that it has only finite numbers of this kind 1, 2, 3, et cetera.”

What is the answer then, from a philosophcal point of view?

The answer lies, of course, in pointing out that the word “finite” is nothing but a negation of the word “infinite”, meaning that we are already by that word invoking a different word-family than the algorithmic word-family (or concept-family). The algorithmic part, one, one plus one, one plus one plus one, doesn't speak about itself. It doesn't say, "finite"; nor does it speak about the "infinite", at least not in the normal context. So when you say "finite" you are as if standing over the process, a perfect filter in hand, a filter that has been shaped by your magnificent mind which is able to wade through a set of infinitely many members and determine what in it is finite and what is infinite. How do you do that? Where is your essential definition of finite? You cannot define "finite" by referring to N, because now we tried to define N by referring to "finite".

No, there is no mistake in the pointing out that there is no clear idea associated with the whole package of all finite positive whole numbers. There is a contradiction at the level of ideas. The reason this contradiction doesn't rip apart every formalism that builds on it all the time is that people are getting used to avoiding infinity except through certain passage-ways that have proven safe. Goedel's Second Incompleteness theorem outlines such safe pathways. It doesn't solve the essential unclarity that lies at the lack of philosophical insight in the very set idea as applied to 'all finite whole positive numbers, and only them.'

We cannot philosophically justify the approach of looking away from some things, at the level of foundational definitions. Attention is the engine of truth. There is a lack of truth in the conception of "natural numbers" and this lack of truth propagates into every field that it contaminates. If mathematics is a giant program, it is a giant program with an essential fierce flaw. It is only a certain type of luck that this flaw hasn't stopped the project, as yet.

The solution to the flaw is to consider that self-reference is inherent in our number concept, as soon as we even vaguely being to talk about 'any' or 'every', or 'etc', or anything like it. This self-reference doesn't require Goedel's smartness to perceive. It takes looking at what we have got, and what we haven't got. We have got a collection of numbers that at every point can refer to its own collection. We haven't got a collection of numbers in which self-referring infinities are excluded. Self-reference is essential existing in the very number concept. And that means that infinity will creep as wanted or (to some) unwanted concept into every complete set.

A PRACTICAL NOTE TO PROFESSIONAL MATHEMATICIANS:

The first part of this note was written in a letter to a friend. It concluded with this advice:

“Don’t let it distract you. The infinity concept, like alcohol and pretty girls, should be drunk in moderation.”

Don’t try to be so true that you don’t get a career. If the world requires you to cheat a little bit, do it. But don’t cheat inside your own mind. You have seen that something is phoney there, at the bottom of mathematics. Be honest enough to yoursel to not try to pretend that what you have seen must be a mistake. You saw the issue, and that seeing has a truth. This truth will likely push you away from the field if you over-focus on it. But if you deny it, it will become a nervousness inside, a sense that you are living on lies.

The infinity concept, exactly because of its self-referential features, is a dangerous one: don’t let it become the sole object in your mind for more than, say, an hour at a time, once every month at most. That may be the uppermost of what a human brain can handle. The rest of the time, do art, if you want to be honest, or something else--something which may require you to lie more, perhaps, to get a profession and get money. Yet you know that there is a truth waiting to be explore, even as a kind of ecstasy, once in a while. Infinity always has positive surprises for us, for those who approach it without hubris.

A.T.